OVERVIEW OF GRADE 8 & 9 PATTERNS, FUNCTIONS AND ALGEBRA

Aspects of an *intended curriculum*

The content and processes of “algebra” in the course *Applications of Dynamic Software* are based on three different, but related views on the nature of algebra, which give meaning and purpose to each problem, activity or task.

A. Algebra as the study of the relationship between variables

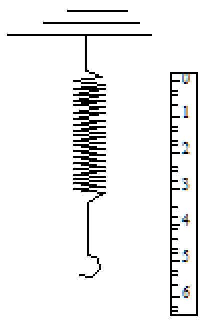
There are many situations involving two variables where the one variable is *dependent* on the other variable, i.e. where a *change* in the value of one (the independent) variable causes a deterministic change in the value of the other (the dependent) variable.

Let us look at a specific example:

The following readings of the mass hung from a spring and the corresponding length of the spring were taken in a scientific experiment.

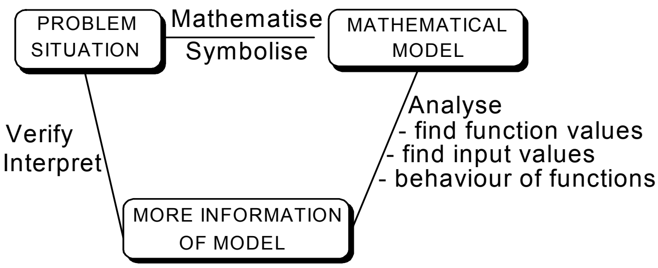
(a) Complete the table.

(b) By how much will the spring stretch if at any stage an additional 2 kg is attached to the spring?



Mass (x kg)	0	1	2	3	4	5	6,6	
Length (y cm)	15	17	19	21	23			32,4

Algebra is a language and a tool to study the nature of the relationship between specific variables in a situation. The power of Algebra is that it provides us with *models* to describe and analyse such situations and that it provides us with the analytical tools to obtain additional, unknown information about the situation. We often need such information as a basis for *reasoning* about problematic situations and as a basis for *decision-making*. This view of algebra can be described as *applied problem solving*.



The main broad objectives of the algebra strand are

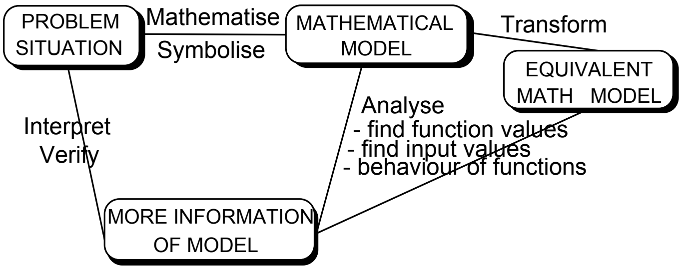
- that learners should experience the *modelling process*,
- that learners should experience and *value* the power of algebraic models to generate new and unknown information, and
- that learners should *understand* the concepts, procedures and techniques involved in modelling.[1]

The mathematical model we construct of the situation may be represented in different ways: in words, as a table of values as above, as a graph or as a computational procedure (e.g. a flow diagram, expression or formula). The additional information we need to generate is mostly of the following three *types*:

1. ***Finding values of the dependent variable (finding function values)***
2. ***Finding values of the independent variable (solving equations)***
3. ***Describing the behaviour of function values (increasing and decreasing functions, rate of change, gradient, derivative, maxima and minima, periodicity, . . .)***

The additional information is obtained by different techniques in different representations of the model (e.g. finding function values by reading from a table, reading from a graph, or substituting into a formula). Some techniques are easier than others and/or yield more accurate results; Therefore, an important aspect of algebraic know-how is *transforming* from one representation of the model (e.g. a table) to another representation (e.g. a formula) which is more convenient to solve problems of the above three types. These transformations are summarised in the following diagram and table:

[1] This must mean understanding its *meaning*, its *significance*, its *logic*, and its *connections* to other ideas



From...to	**Words**	**Table**	**Graph**	**Formula**
Words				
Table				
Graph				
Formula				

In particular, the above three problem types are handled more easily when a formula or function rule is available. In our spring example above, the formula $y = 2x+15$ easily finds the missing data in the table. Therefore finding formulae is important – it is our fourth problem type:

4. Finding a function rule (formula)

Learners should be able to find function rules in different representations of the model, i.e. to find the function rule from words, from a table or from a graph. The processes involved in finding function rules include induction (recognising a pattern in a table of values) and analytical processes (deduction, e.g. solving simultaneous equations). The relationships between tables, graphs and formulae are of particular importance (e.g. the relationship between a recursive common difference of 2 in a table, a gradient of 2 in the graph and the coefficient 2 in the formula $y = 2x + 3$).

Sometimes the formula describing the relationship between variables may be "complex", making the first three problem types above very "complex". In such cases it is convenient to first transform the formula to an *equivalent*, but more convenient form for a specific task (from formula to formula in the previous table). This defines our fifth important problem type:

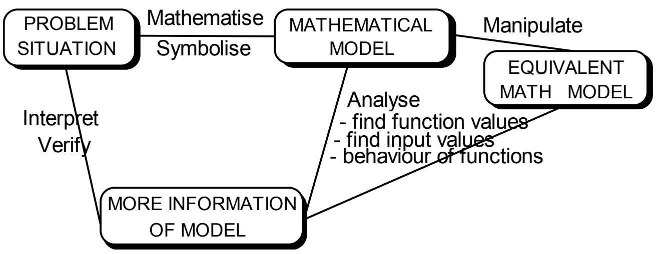
5. Transforming to an equivalent formula ("manipulation" of algebraic expressions)

We make a few brief further remarks on equivalent transformations:
We should beware of superficial interpretations of algebraic manipulation, e.g. that it is "operations on or calculation with *letters*", in the same way as arithmetic involves "calculation with *numbers*". The arithmetical *operations* (addition, multiplication, etc.) are only defined on *numbers* not on *symbols*. A statement like $x(x + 3) = x^2 + 3x$ therefore does not indicate multiplication at all: $x.x = x^2$ and $3 \times x = 3x$ are merely short algebraic *notations*, not the *results* of multiplication. If we replace x with a number, e.g. if $x = 7$, the statement becomes $7(7 + 3) = 7^2.+ 7 \times 3$ and it is clear that we did not multiply or calculate in the usual arithmetic way of saying $7 \times 3 = 21$. Rather, each of the statements $7(7 + 3) = 7^2 + 7 \times 3$ and $x(x + 3) = x^2 + 3x$ represents *equivalent* computational *procedures*, the right-hand side is not the *answer* of the left-hand side. For example, if a netball team has a party and the food costs R7 and the transport costs R3, then $7(7 + 3)$ and $7\times7 + 7\times3$ merely describe two different methods of calculating the total cost, both yielding the same numerical answer. The difference between the processes in arithmetic and algebra should be clear from the following example:

	Arithmetic	**Process**		**Algebra**
	12×17			
=	$12(10 + 7)$	*renaming the numbers*		$x(x + 3)$
=	$12 \times 10 + 12 \times 7$	*equivalent transformation*	=	$x^2 + 3x$
=	$120 + 84$	*sub-calculations*		
=	204	*final calculation*		

Algebraic "manipulation" involves *equivalent transformations:* $x(x + 3)$ and $x^2 + 3x$ represent two *different computational procedures*, and we may in a particular context choose any of the two because they are the *same* in the sense that they yield the same *values* for the same values of *x:*

x	1	2	3	4	5
$x(x + 3)$	4	10	18	28	40
$x^2 + 3x$	4	10	18	28	40



Algebraic manipulation is a useful tool in problem solving, but should not be elevated to a skill for its own sake. Therefore algebraic manipulation should be developed and practised in the context of equivalent transformations, as illustrated in the modelling diagram above. Our teaching should develop the meaning of equivalent expressions and students should *experience the usefulness* of such transformations to make the first three problem types above easier. For example, consider these two problems:

If $x = 7{,}3$ evaluate $\dfrac{4x^2 + 2x}{2x}$

Solve for x if $\dfrac{4x^2 + 2x}{2x} = 11$

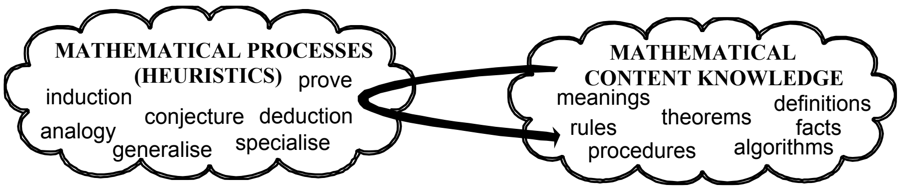
As they stand they are difficult. However, they become fairly easy once we transform $\dfrac{4x^2 + 2x}{2x}$ to $2x + 1$.

In problem solving, the choice of model depends on the properties of the model and the characteristics of the situation. In studying the relationship between variables, it is therefore important to analyse the different behaviours (i.e. properties) of different models (functions). In the junior secondary phase we study simple direct proportion (y = kx), the linear function (y = mx + c) and inverse proportion (xy = k), while quadratic, exponential, logarithmic and trigonometric functions are studied in the senior secondary phase.

It is important that the teaching programme provides for appropriate experiences of all the problem-types and that it develops the underlying concepts and techniques to enable learners to experience the power of algebra as a tool to solve problems. However, it should be emphasised that our objective should be to *solve problems*, not to master isolated skills for its own sake (say *factorisation*).

B. Algebra as generalised arithmetic

The view that, historically, algebra grew out of arithmetic, and ought so to grow afresh for each individual, has much to say about the nature of Algebra and about the teaching and learning of Algebra. Algebra as generalised arithmetic generates "new" mathematical knowledge from existing mathematics through important mathematical *processes* such as induction, deduction, generalisation and proof.



Algebra as generalised arithmetic can involve simple numerical situations such as

Investigate the sum of three consecutive numbers.

An inductive approach (taking a number of special cases) may yield:

1 + 2 + 3 = 6
2 + 3 + 4 = 9
3 + 4 + 5 = 12
4 + 5 + 6 = 15

Most learners will easily recognise (*abstract*) the pattern as multiples of 3 and *generalise* by *conjecturing* that the sum is always a multiple of 3. However, learners should have adequate experience of the pitfalls of induction[2] to realise that it is necessary to *prove* the validity of the conjecture or to *explain* its form. This requires the introduction of a "generalised number" n to cater for *any* natural number, indeed, for *all* natural numbers and *deductive* reasoning:

$$n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1)$$

From this *form*, we should recognise that $3(n + 1)$ is a multiple of 3 for *any* value of n. From this form we can also deduce that the sum is always three times the middle number.

Seeing algebra as generalised arithmetic opens the opportunity to engage with rich problems in *Number Theory* (e.g. multiples, remainders, prime numbers, powers, etc.) with different meanings for algebraic expression, e.g. $3n$ is a multiple of 3 for *all* $n \in$ N and $3n + 2$ leaves a remainder of 2 when divided by 3.

[2] For example, it is enticing to think that because $n^2 - n + 11$ yields primes for values of n from 1 to 10, that the value is prime for *all* $n \in$ N. But it is not prime for $n = 11$!

C. Algebra as a study of procedures to solve word problems

Here is a very simple example:

When 3 is added to 5 times a certain number, the sum is 40. Find the number.

The problem is easily translated into the language of algebra, yielding an *equation*, with x as *unknown:*

$$5x + 3 = 40$$

To solve the problem we need to solve the equation using techniques like reversed flow diagrams and *equivalent equations*:

$$\Leftrightarrow\ 5x + 3 - 3 = 40 - 3$$

$$\Leftrightarrow 5x = 37, \text{ and then } x = 7{,}2$$

Three different objects

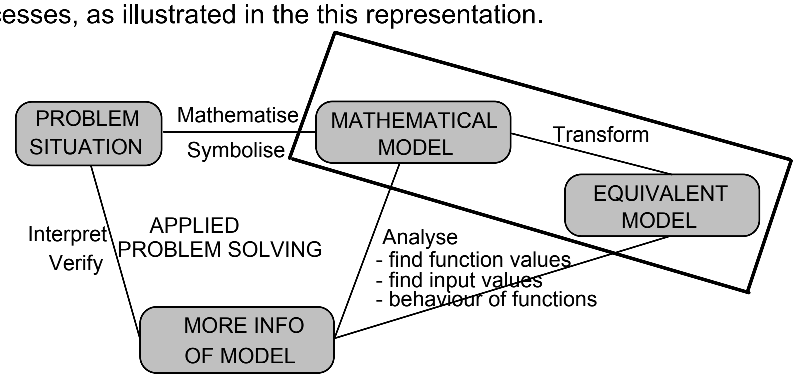
Our three views (conceptions) of Algebra lead to three quite different underlying mathematical objects. Generalised arithmetic often deals with *identities* such as $x(x + 3) = x^2 + 3x$. Algebra as a study of relationships between variables deals mostly with *formula*e. All three views of Algebra involve *equations* as special cases (problem type 2, finding the input number x for a given output number $y = 40$). Formulae, identities and equations are three different types of *objects* in algebra, defining different meanings of letter symbols and tasks as summarised below.

Context	Examples	Meaning of x *and* y
Equation	$3x + 2 = 5$	An *unknown* – a number to be found that will make the sentence true
Identity	$3x + 2x = 5x$ $x + y = y + x$	A *generalised number* – *any* value of x (and y) makes the sentence true. x and y are *unspecified* numbers and y is independent of x.
Formula	$y = 3x + 2$	A *variable* – x is *any* number within the restriction of a physical situation. The value of y *depends* on the value of x.

Remarks on teaching

For children to make sense of mathematics and to understand mathematics, they must for each mathematical object (e.g. variable, identity) and process (e.g. factorisation) understand its *meaning*, *significance* (its usefulness) and its *logic* (is it true, why is it true?).

Traditional teaching of algebra emphasises "manipulation", i.e. the transformation of algebraic expressions to more convenient equivalent expressions. However, such manipulation is done mainly in isolation of the other important processes, as illustrated in the this representation.



In doing this, we have elevated manipulation to an objective in its own right, to be mastered for its own sake and evaluated on its own. The emphasis was therefore on "factorise", "simplify", . . . In placing the emphasis on applied problem solving (relationship between variables, formulas), mathematical problem solving (generalised arithmetic, identities) and word problems (equations), we emphasise that manipulation is but one of the important processes in the service of problem solving.